Maths for Economics

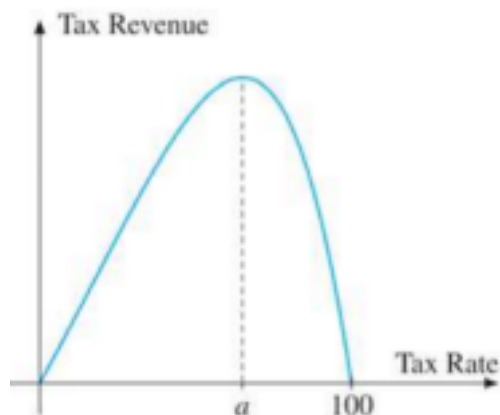# 5. Functions

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1 Functions

Some of the most useful mathematical tools have to do with analyzing various functions.

1.1 Definitions

A function is a relationship that takes an input variable and gives back a *single* output variable. The total amount of money in a bank account that started with a $100 deposit and a 3% interest rate is a function of the time since the deposit was made. The amount of a product that will be sold is a function of the price. The number of mortgages that will be initiated is a function of the interest rate.

Functions can also take multiple inputs, but they can always give back only one output. To keep things simple we're going to work with functions that take only one input as well.

Functions can be graphed on the Cartesian plane. One example is the Laffer curve, which is a function that takes a tax rate and gives back a total amount of tax revenue that will be generated at that tax rate:

We usually name functions with lowercase letters, just like numbers, to reflect the fact that the value of a function is a single number. Sometimes parentheses are used to specify which number/variable is the input. Here are some examples:

- $y = 2x$: This function (y) takes an input number x and returns its double.

- $f(x) = \sqrt{x} + 2$: This function f takes an input number x and returns its square root, plus 2. Notice that the function won't be able to be calculated if $x < 0$. That's ok - functions don't have to be defined on every possible input value. The **domain** of a function is the set of numbers for which the function is defined. In this case, the domain is $[0, \infty)$.

- $f(x) = 3^x$: This function raises 3 to whatever power the input variable is. The graph below shows what this function looks like. Notice that no matter what x is, the output value never gets down to 0. The **range** of a function is the set of values that the function *might* produce, and in this case the range is $(0, \infty)$.
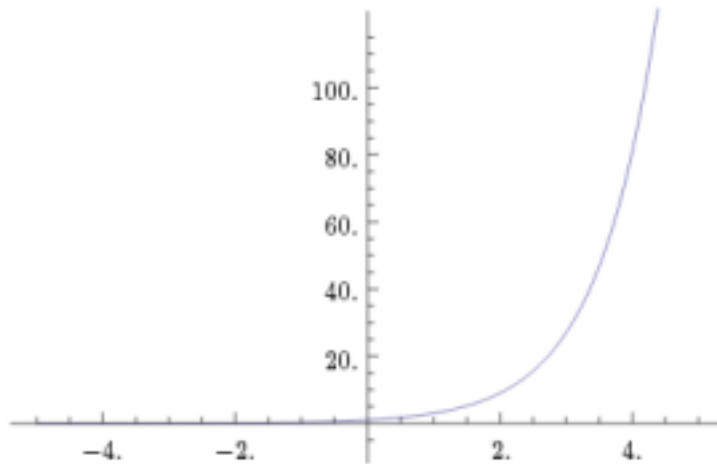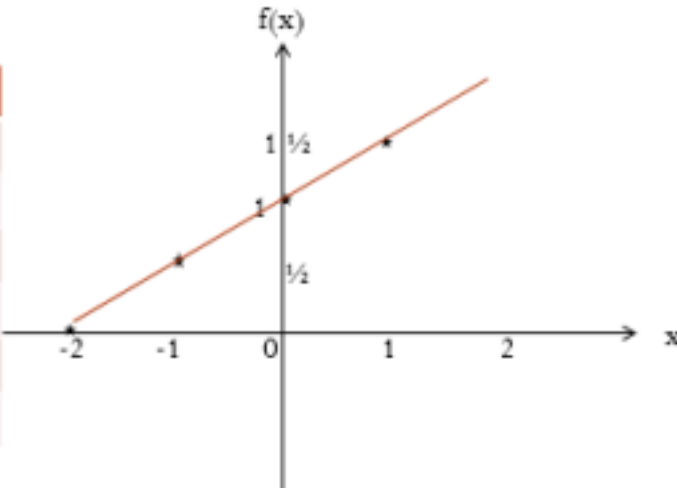

1.2 Linear functions

Linear functions are one of the simplest relationships, but don't underestimate their power. Linear relationships are all around us in the economy, and so it's important to understand how to work with them.

1.2.1 Definition

A linear function is one that looks like a line when we plot it in the Cartesian plane. They have a function definition of the form $f(x) = ax + b$, where a and b are called the **slope** and **intercept** respectively. We can see why if we plot an example:

$$f(x) = \frac{x}{2} + 1$$

x	f(x)
-2	0
-1	½
0	1
1	3/2
2	2
3	5/2

In this example, $a = \frac{1}{2}$ is the slope of the line in the graph: For each unit we move right, the graph goes up by $\frac{1}{2}$ of a unit. $b = 1$ is the intercept, which is where the line intersects the vertical axis of the plot.

Linear functions can be frequently used to understand economic relationships. For example, say it costs \$182,100,000 to build a factory for hard drives. Once it's built, each hard drive costs \$55.73 to produce. The total cost function for the hard drive manufacturer is given by the linear function $C(x) = 182,100,100 + 55.73x$, where x is the number of hard drives produced.
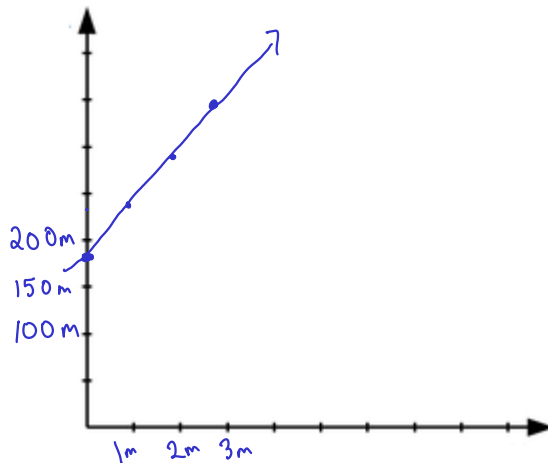
$C(1000000) = 182.1 + 55.73$ million
$= 235.83$ million

$C(2000000) = 182.1 + 55.73 \cdot 2$
$= 291.56$

etc.

1.2.2 Finding a linear relationship

Any two points can have a line drawn between them, and so any two points is enough to define a linear function. How do we do that? The linear function equation has two key components that we have to figure out from the information we have: the slope and the

intercept. If we have two points, we can figure out the slope by calculating how much the line goes up between those two points and dividing by the horizontal distance between them.

Say we have two points on a line, $(4, 3)$ and $(-2, -1)$. The horizontal distance between these two points is 6, and the vertical distance between them is 4, so the slope is $4/6 = 2/3$.
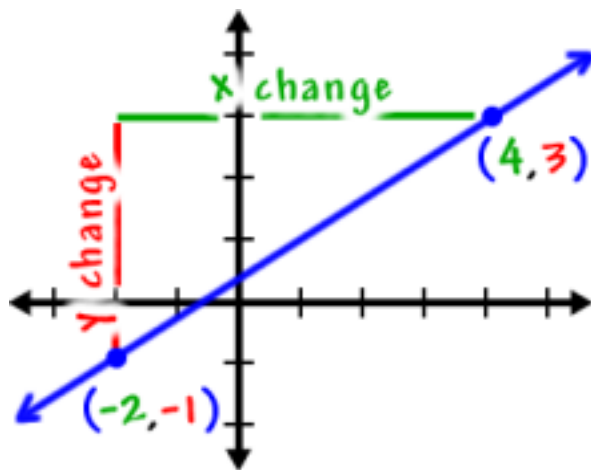
Therefore we know the equation of the line is $f(x) = \frac{2}{3}x + b$, for some number b that we don't know yet.

We can find b by realizing that it has to be the value that forces the function to output the points that we already know. So when we input -2 we should get -1 back, or if we input 4 we should get 3 back.

$$f(x) = \frac{2}{3}x + b \Rightarrow 3 = 4\frac{2}{3} + b \Rightarrow b = \frac{1}{3}.$$

Of course, it doesn't matter which point we use to anchor the value of b:

$$f(x) = \frac{2}{3}x + b \Rightarrow -1 = -2\frac{2}{3} + b \Rightarrow b = \frac{1}{3}.$$

The final equation is therefore

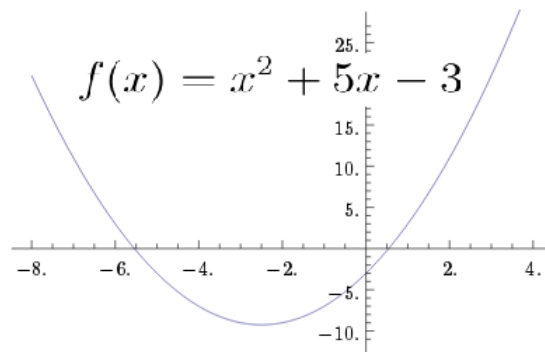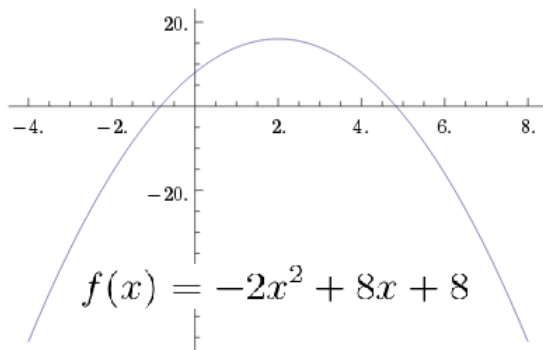$$f(x) = \frac{2}{3}x + \frac{1}{3}.$$

1.3 Quadratic functions

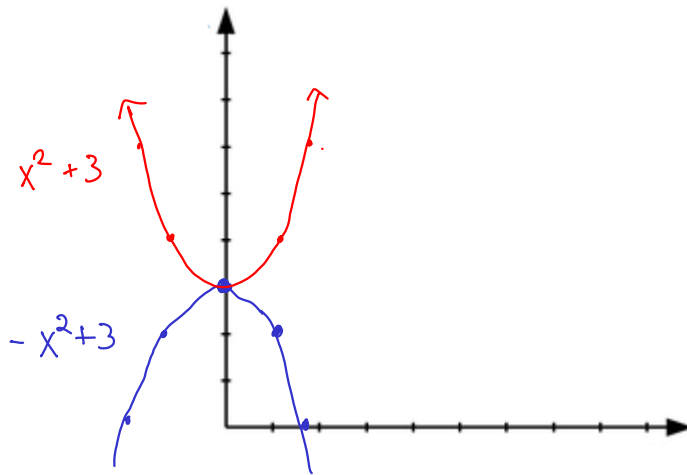A quadratic function adds one term to a linear function.

1.3.1 Definition

The general form is

$$f(x) = ax^2 + bx + c$$

and these functions look like either a bowl or a hill:

$$f(x) = x^2 + 5x - 3$$

$$f(x) = -2x^2 + 8x + 8$$

The left hand scenario, with a hill shape, occurs when $a < 0$; the bowl shape occurs when $a > 0$.

$x^2 + 3$

$-x^2 + 3$

1.3.2 Vertex

Quadratic functions can be used to model many economic phenomena as well. The Laffer function, for example, could be approximated with a quadratic function. In that example, we might be interested in figuring out which tax rate would maximize government revenue. There is luckily an easy formula to identify the input value at the vertex of the bowl or hill of the quadratic function.

$$\text{Vertex value of } x = -\frac{b}{2a}$$

We can then of course input this x value into the function to find out what the maximum or minimum value is.

Let's use the examples in the graphs above. In the first case, $f(x) = -2x^2 + 8x + 8$, which has $b = 8$ and $a = -2$, and so the vertex occurs at $-\frac{b}{2a} = -\frac{8}{2 \cdot (-2)} = 2$, which matches what the graph shows. The maximum value of the function is the output of the function at this value of x: $f(2) = -2 \cdot 2^2 + 8 \cdot 2 + 8 = 16$.

In the second example, $f(x) = x^2 + 5x - 3$, so $a = 1$ and $b = 5$. The minimum point must occur where $x = -\frac{b}{2a} = -\frac{5}{2}$. The minimum value is thus $f\left(-\frac{5}{2}\right) = \left(-\frac{5}{2}\right)^2 - 5\frac{5}{2} - 3 = -\frac{37}{4}$.
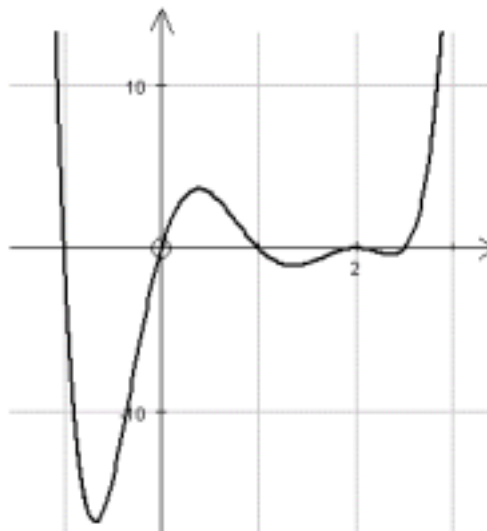
1.4   Polynomial functions

Linear and quadratic functions are types of polynomials, but polynomials generally can have even more terms with x^n in them, for any $n \in \mathbb{N}$. Here are some examples:
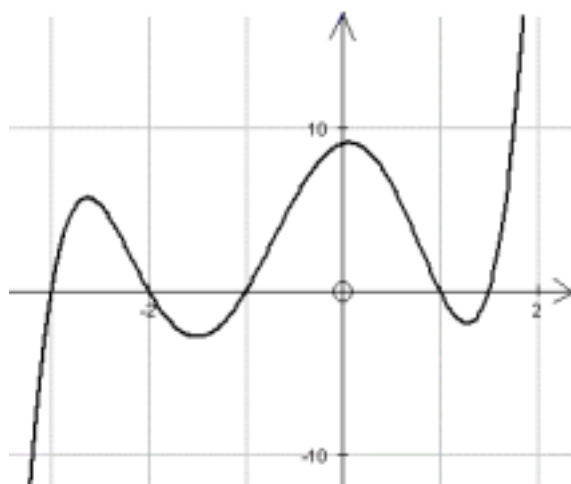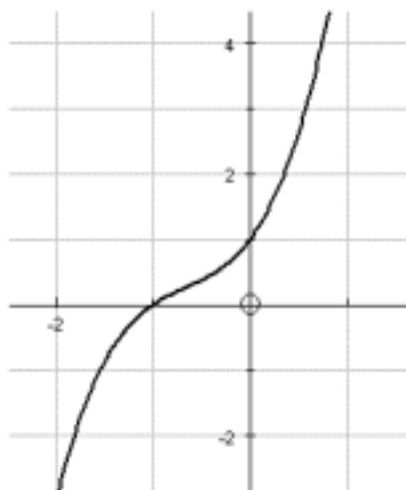
$$f(x) = x^4 - 2x^2$$
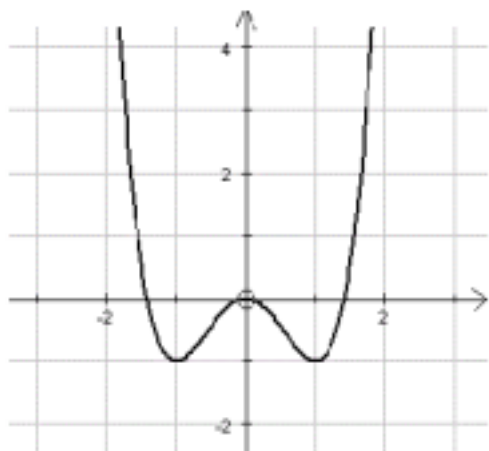
$$f(x) = 2x^6 - 13x^5 + 26x^4 - 7x^3 - 28x^2 + 20x$$

$$f(x) = x^5 + 3.5x^4 - 2.5x^3 - 12.5x^2 + 1.5x + 9$$
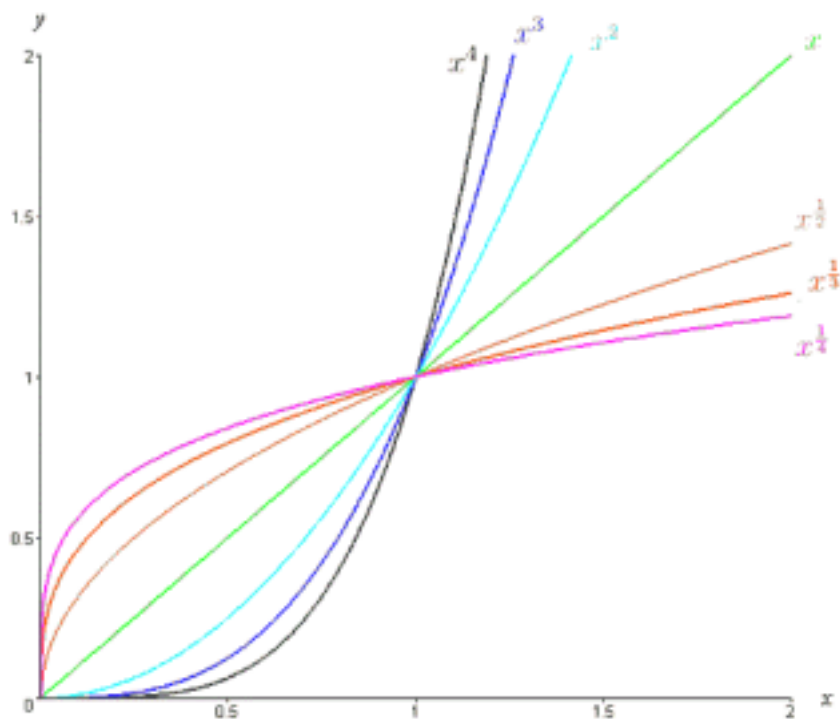

$$f(x) = x^3 + 2x^2 + 2x + 1$$

1.5 Power functions

While polynomials only have natural numbers in the exponents of the input variable, we can also define functions with fractional powers. We call a function a **power function** if it takes the form

$$f(x) = ax^r$$

where $r > 0 \in \mathbb{R}$.

When the exponent is $r > 1$, the function curves upwards. This kind of function can be used to model a production function with increasing returns to scale; for example, an company that produces specialized computer chips might find it very expensive to start production in the first place. But the more they invest in their factory, the more efficient machinery they can purchase to speed up their production line. These capital investments increase their overall output more than proportionally.

When the exponent is $r < 1$, the function flattens out over time. This might represent production of an oil company. Initially oil extraction is easy and cheap, but the company must then switch to more expensive methods like shale oil extraction. Production increases less than proportionally with costs.

1.6   Exponential functions

Exponential functions put the the input variable of the function in the exponent itself. These functions take the form

$$f(x) = a \cdot b^x.$$

The most common exponential function used in economics is $f(x) = e^x$. e is a special number that real but not rational, and it pops up naturally in many economic processes. Additionally, it's also very convenient to work with mathematically. We'll learn more about why at various points in the course. For now, let's just define $e = 2.71828182846...$ (which, since e is not rational, continues on forever). Here's what this function looks like:
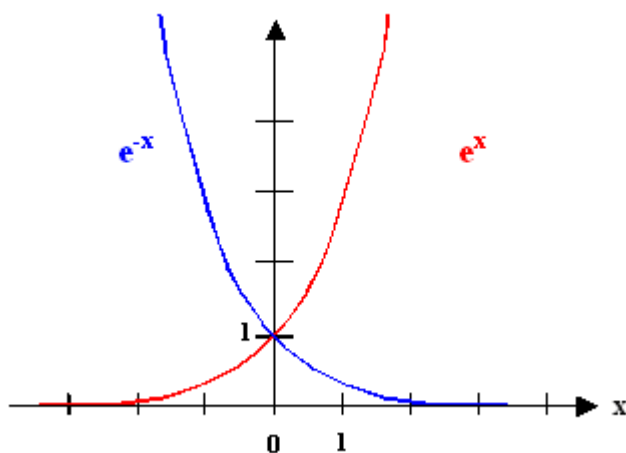
The graph above also shows a common variant, $f(x) = e^{-x}$.

1.7   Logarithmic functions

The opposite of an exponential function is a logarithmic function. While a^b describes the result of multiplying a by itself b times, $\log_a x$ asks what power a needs to be raised to in order to reach x. In this expression, a is called the **a** of the logarithm.

$$\log_a x = b \Leftrightarrow a^b = x$$

For example, $\log_2 8 = 3$, $\log_1 0100 = 2$, and $log_e 1 = 0$.

The most common base we use is e, for the same reasons that e is the most common base of exponential functions that we'll use. In fact, it's so common that economists often don't write it. In other fields, a different base is the default base and $\log_e x$ is written as $\ln x$, but in economics, both $\ln x$ and $\log x$ mean $\log_e x$. The logarithm function with base e is called the **natural logarithm**.
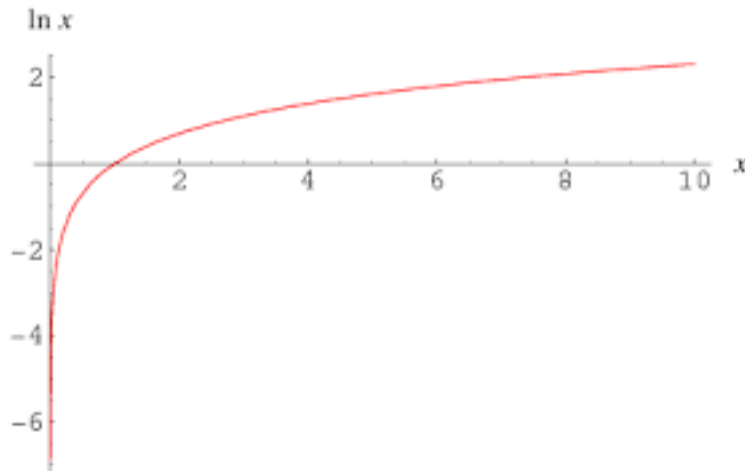
The logarithm function has some convenient properties. You can try to convince yourself that these are true:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

for any positive value, $non-one$ value of c. This is a convenient way to convert logarithms into the natural logarithm with base e:

$$\log_a b = \frac{\log b}{\log a}$$

Other rules for working with logarithms:

$$\log_b (xy) = \log_b x + \log_b y$$

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b (x^y) = y \cdot \log_b x$$

2 Exercises

1. Write down functions for the following relationships to answer the question (there may be multiple possible answers):

   (a) 8 workers can complete inventory in 6 hours. How many must be hired to complete the job in 4 hours?

   48 manhours.
   $y = $ workers, $x = $ time. $y = \frac{48}{x}$ $\Rightarrow$ $y(4) = 12$.

   (b) $100 is put into a bank account that earns 3% per year. How much will the account contain in 2 years?

   $y = \$$, $x = $ years. $y = 100 \cdot 1.03^x$ $\Rightarrow$ $y(2) = 106.09$

   (c) $100 is put into a bank account, and after 3 years the account contains $120. What is the annual interest rate of the account?

   $x = $ rate, $y = $ amount.
   $y = 100 (1+x)^3$. $120 = 100 (1+x)^3$ $\Rightarrow$ $x = 6.27\%$

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(d) Chocolate costs \$5 per kilogram and milk costs \$3 per kilogram. How much per kilogram does chocolate milk containing 5% chocolate cost?

$C = \text{cost}, \quad x = \text{fraction chocolate}.$

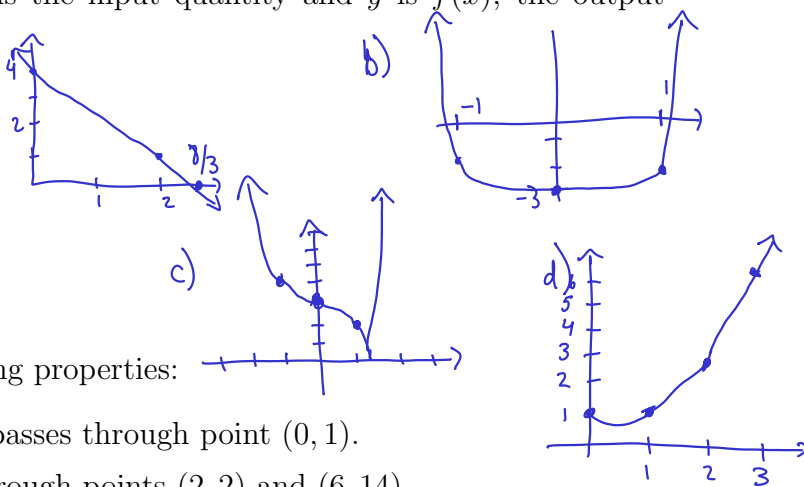$C = x \cdot 5 + (1-x)3 \quad \Rightarrow \quad c(.05) = .25 + 2.88 = 3.13$

2. Graph the following functions. x is the input quantity and y is $f(x)$, the output quantity.

(a) $3x + 2y = 8 \quad \Rightarrow \quad y = 4 - \frac{3}{2}x$

(b) $y = x^4 - 3$

(c) $f(x) = |x^3 - 3|$

(d) $f(x) = 2^x - \sqrt{x}$

3. Find the functions with the following properties:

$y = 1 + 2x$ (a) Linear function with slope 2, passes through point $(0,1)$.

$y = 3x - 4$ (b) Linear function that passes through points $(2,2)$ and $(6,14)$.

(c) Linear function that passes through $(0,0)$. Parallel to the line through $(2,3)$ and $(-1,6)$. $\quad y = -x$

$y = x^2 - 2x$ (d) Quadratic function with $a = 1$ and with vertex at $x = 1$. Passes through $(0,0)$.

$y = \frac{1}{2}x^3$ (e) Power function with $a = \frac{1}{2}$. Passes through $(2,4)$.

4. Solve the following equations for x, using either algebra or a graph:

(a) $8^{2x-3} - 4 = 5 \quad \Rightarrow \quad 8^{2x-3} = 9 \quad \Rightarrow \quad \log 8^{2x-3} = \log 9 \quad \Rightarrow \quad (2x-3)\log 8 = \log 9 \quad \Rightarrow \quad 2x-3 = \frac{\log 9}{\log 8}$

$\Rightarrow \quad 2x = 3 + \frac{\log 9}{\log 8} \quad \Rightarrow \quad x = \frac{3}{2} + \frac{\log 9}{2\log 8}$

(b) $\log 2 + 2\log x = \log(7x - 3)$

$\Rightarrow \log 2 + \log x^2 = \log(7x-3)$

$\Rightarrow \log(2x^2) = \log(7x-3)$

$\Rightarrow 2x^2 = 7x - 3 \quad \Rightarrow \quad 2x^2 - 7x + 3 = 0$

$\Rightarrow (2x - 1)(x - 3) = 0$

$\Rightarrow x = \frac{1}{2} \text{ or } 3$

(c) $\left(\frac{4}{3}\right)^{1-x} = 5^x$

$\log\left(\frac{4}{3}\right)^{1-x} = \log 5^x$

$\Rightarrow (1-x)\log\left(\frac{4}{3}\right) = x \log 5$

$\Rightarrow \log\frac{4}{3} - x\log\frac{4}{3} = x\log 5$

$\Rightarrow \log\frac{4}{3} = x\left(\log 5 + \log\frac{4}{3}\right)$

$\Rightarrow \frac{\log 4 - \log 3}{\log 5 + \log 4 - \log 3} = x$

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